The Bernoulli equation
and
the Kelvin circulation theorem

Conditions for Bernoulli equation:

- Stationary system: $\partial \ldots / \partial t = 0$
- Isoentropic motion: $ds/dt = 0$

Introduce enthalpy w: thermodynamic potential:

$$w = \epsilon + PV = \epsilon + \frac{P}{\rho}, \tag{1}$$

where P and ρ are gas pressure and density. Differential of w is

$$dw = Tds + \frac{dP}{\rho}. \tag{2}$$

Thus, for $ds = 0$ we have

$$dw = \frac{dP}{\rho}. \tag{3}$$

Write the Euler equation assuming for a moment that there is no gravity. It will be easy to add gravity at the very end:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla P}{\rho}. \tag{4}$$

Now use enthalpy:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\nabla w. \tag{5}$$

A useful relation from vector analysis:

$$\frac{1}{2}\text{grad}\, v^2 = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{v}\nabla)\vec{v}. \tag{6}$$

Using this relation we can re-write the Euler equation in this pleasant form:

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\text{grad}\,(w + \frac{v^2}{2}). \tag{7}$$

Now we can add the gravity. If ϕ is the gravitational potential, then

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\text{grad}\,(w + \frac{v^2}{2} + \phi). \tag{8}$$

For a stationary system the first term in the l.h.s is zero:

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \text{grad}\,(w + \frac{v^2}{2} + \phi). \tag{9}$$

Introduce a new notion: streamlines. A streamline is a curve in space such that it is tangent to the velocity vector $\vec{v}(x, y, z)$ at any point of space (x, y, z). By definition, the equations defining the streamlines are:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}. \tag{10}$$

Let's define a unit vector $\vec{l}(x, y, z)$, which is tangent to the streamline at any point. It is parallel to velocity $\vec{v}$. Multiply equation (9) by $\vec{l}$. Because for any vectors $\vec{v} \times \vec{a} \perp \vec{v}$, the scalar product $\vec{l} \cdot \vec{v} \times \vec{\nabla} \vec{v} = 0$. Thus, the l.h.s. is equal to zero. On the r.h.s. we have a scalar product of $\vec{l}$ and a grad of a function, which is a projection of the gradient in the direction of the streamline. In turn, it is a derivative of that function along the direction of streamline:

$$\vec{l} \cdot \text{grad} \left(w + \frac{v^2}{2} + \phi \right) = \frac{\partial}{\partial l} \left(w + \frac{v^2}{2} + \phi \right) = 0. \tag{11}$$

Thus, along any streamline the Euler equation is reduced to:

$$w + \frac{v^2}{2} + \phi = 0. \tag{12}$$

Replacing enthalpy w with its expression through the internal energy ϵ, we get the Bernoulli equation:

$$\boxed{\epsilon + \frac{P}{\rho} + \frac{v^2}{2} + \phi = 0.} \tag{13}$$

Conditions for Kelvin circulation theorem:

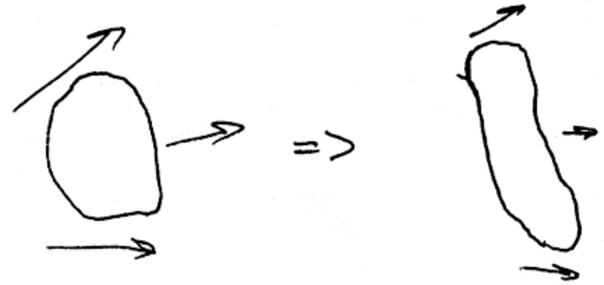
- Isoentropic motion: $ds/dt = o$

Take a contour in a fluid. The contour is made of fluid elements. Position of the contour changes with time as the elements drift with the fluid motion. Take a "circulation" along the contour:

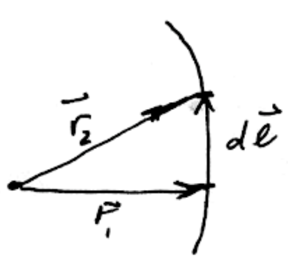
$$\Gamma = \oint \vec{v} d\vec{l} \tag{14}$$

How does it change with time? Consider line element $d\vec{l}$ of the contour: $d\vec{l} = \vec{r}_2 - \vec{r}_1 = \delta r$, where $\vec{r}_2$ and $\vec{r}_2$ are radius vectors fo the end and the beginning of $d\vec{l}$.

Take a contour inside the fluid. Position of the contour changes with time as particles, which mark the contour, move with the fluid



How $\Gamma = \oint \vec{v} d\vec{l}$ changes with time?



$d\vec{l} = \vec{r}_2 - \vec{r}_1 = \delta\vec{r}$, where $\vec{r}_2$ and $\vec{r}_1$ are vectors of the end and the start of $d\vec{l}$

$$\frac{d}{dt}\oint \vec{v}\,\delta\vec{r} = \oint \frac{d\vec{v}}{dt}\cdot\delta\vec{r} + \oint \vec{v}\,\frac{d}{dt}\delta\vec{r} \qquad (*)$$

$$\vec{v}\cdot\frac{d}{dt}\delta\vec{r} = \vec{v}\cdot\frac{d}{dt}(\vec{r}_2 - \vec{r}_1) = \vec{v}\cdot(\vec{v}_2 - \vec{v}_1) = \vec{v}\cdot\delta\vec{v} = \delta\left(\frac{v^2}{2}\right)$$

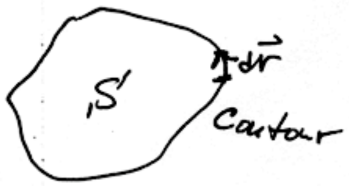
For a closed contour $\oint \delta f = 0$, thus the second integr in (*) is zero

For isoentropic motion the Euler equation has the form

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v} = -\nabla w, \quad w = \varepsilon + \frac{p}{\rho}$$

or

$$\frac{d\vec{v}}{dt} = -\nabla w$$



Stokes theorem:

$$\oint_{\text{contour}} \vec{F}(\vec{r})\cdot d\vec{r} = \oint_{\text{surface } S} d\vec{S}\cdot(\vec{\nabla}\times\vec{F})$$

Using the Stokes theorem we can rewrite the last integral in (*)

$$\oint \frac{d\vec{v}}{dt}\cdot d\vec{r} = \oint d\vec{S}\cdot\nabla\times\frac{d\vec{v}}{dt} = \oint d\vec{S}\cdot\nabla\times\nabla w = 0$$

curl ≡ rot ≡ ≡ $\nabla\times$

Because curl · grad $w = 0$

Thus

$$\oint \vec{v}\cdot d\vec{l} = \text{const}$$

The only assumption made was isoentropic motion. It can be shown, that the Kelvin theorem holds if we include gravity.

Potential flow

If we choose an infinitisimally small contour

$\bigcirc \Rightarrow d\vec{S}$

$$\oint \vec{v}\,d\vec{l} = \oint \text{curl}\,\vec{v}\,d\vec{S} \approx \text{curl}\,\vec{v}\cdot d\vec{S} = \text{const}$$

(do not mix with w (enthalpy))

$\vec{\omega} = \text{curl}\,\vec{v}$ is called vorticity

For isoentropic flow vorticity is preserved : $\vec{\omega} = \text{const}$

Spin Parameter and origin of angular momentum of galaxies

- $L =$ angular momentum
- $M =$ total mass
- $E =$ total energy

We define dimentionless spin parameter λ as a ratioof the frequency of rotation ω to the frequency of rotation $\omega_{\rm sup}$ of the object if it were rotationally supported:

$$\lambda = \frac{\omega}{\omega_{\rm sup}}, \tag{15}$$

$$\omega = \frac{L}{MR^2}, \tag{16}$$

$$\omega_{\rm sup} = \sqrt{\frac{GM}{R^3}}, \quad \omega_{\rm sup}^2 R = GM/R^2) \tag{17}$$

$$\tag{18}$$

Taking $E \approx GM^2/R$, we get:

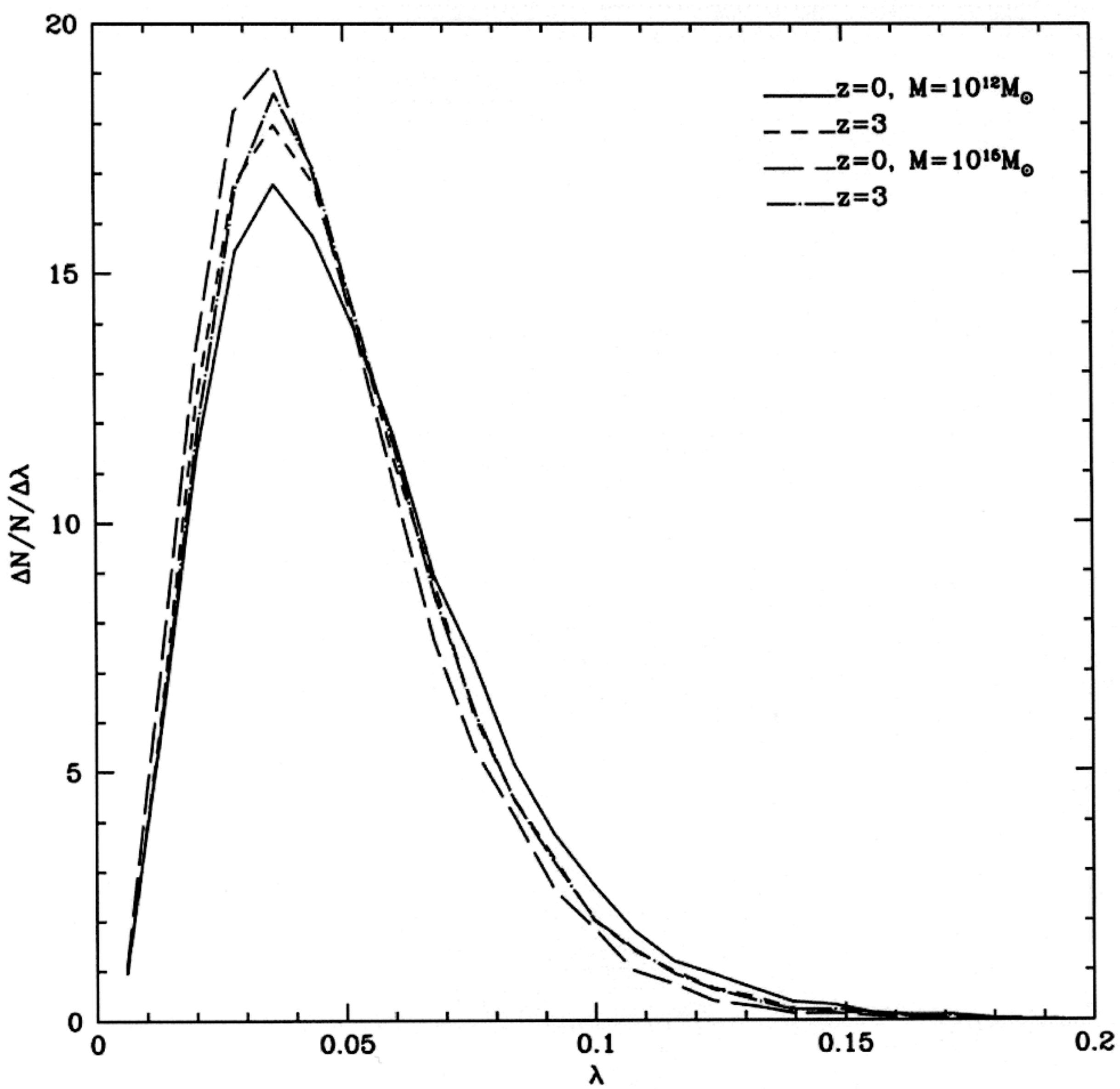
$$\lambda = \frac{\omega}{\omega_{\rm sup}} = \frac{LE^{1/2}}{GM^{5/2}}. \tag{19}$$

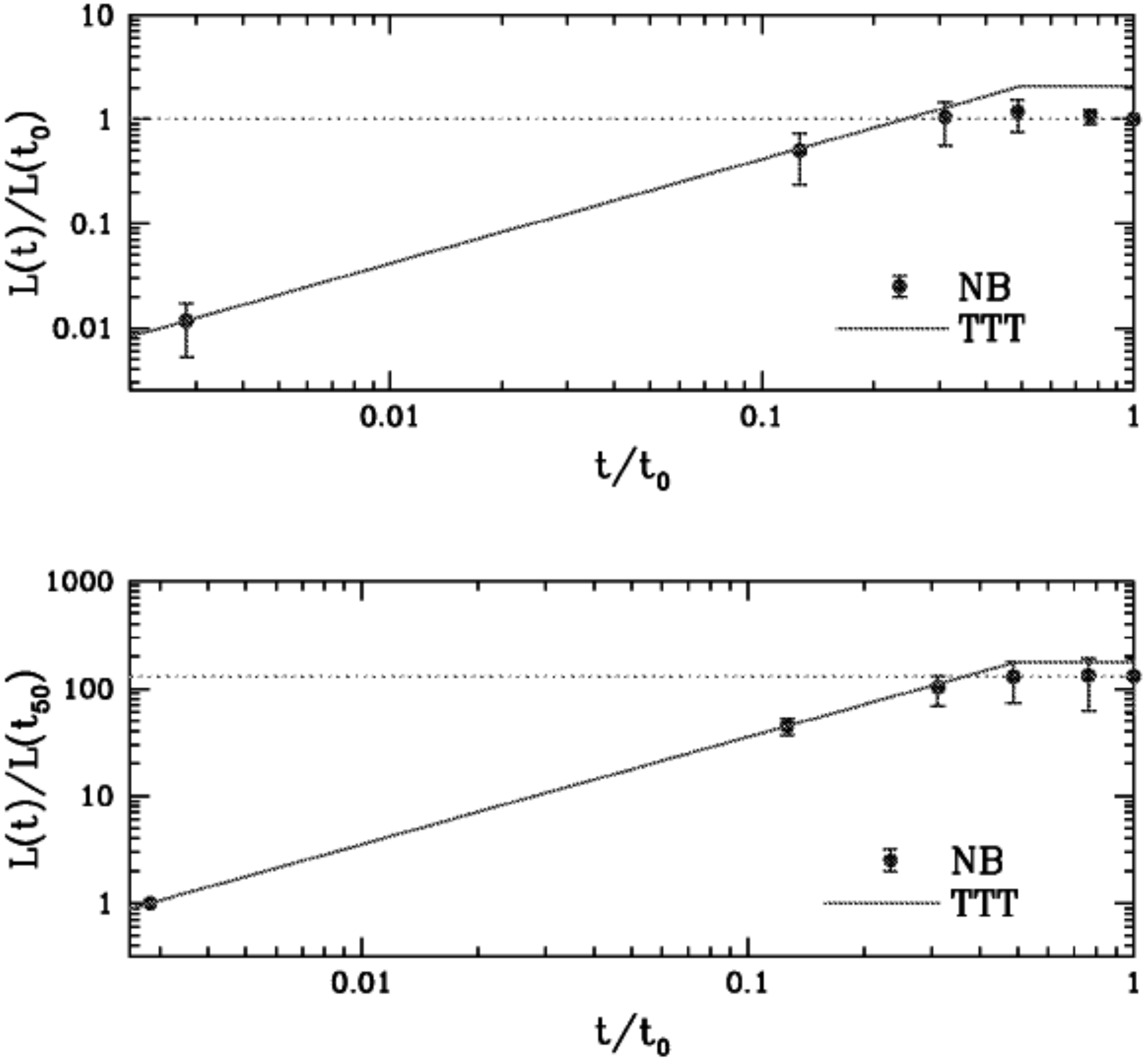
Dark matter halos in cosmological simulations have very little rotation: $\lambda \approx 0.05$. This implies that gas must lose a large fraction of its energy before it settles and forms a disk of a spiral galaxy.

$$p(\lambda)d\lambda = \frac{1}{\sigma_\lambda\sqrt{2\pi}}\exp\left(-\frac{\ln^2(\lambda/\bar{\lambda})}{2\sigma_\lambda^2}\right)\frac{d\lambda}{\lambda}. \quad (2)$$

The parameters for the log-normal distribution were found to be $0.03 \leq \bar{\lambda} \leq 0.05$ and $0.5 \leq \sigma_\lambda \leq 0.7$ for standard CDM and various variants (e.g. Warren et al. 1992; Gardner 2001). For the ΛCDM cosmology with matter density $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$, and $\sigma_8 = 1$, the log-normal parameters were found to be $\bar{\lambda} = 0.042 \pm 0.006$ and $\sigma_\lambda = 0.50 \pm 0.04$ (Bullock et al. 2001b). Note that the $\bar{\lambda}$ parameter of the log-normal distribution (2) is not equal to the mean of λ; rather, $\langle\lambda\rangle \approx 1.078\bar{\lambda}$ for $\sigma_\lambda = 0.5 - 0.6$.

Distribution of the spin parameter for halos at $z = 0$ and 3, for halos whose main progenitor has mass 10^{12} or 10^{15} $M_\odot$ at redshift $z = 0$. The spin distribution has rather weak dependence on mass and redshift.





Evolution of angular momentum L(t) of a patch of dark matter over time.

TTT = predictions of the tidal torque theory
NB = N-body simulations

Figure 5. True evolution of spin amplitude in the simulation compared to TTT. Shown are the average and the 68.3 per cent confidence interval over the haloes at different times, of the ratio of spin amplitude $L(t)$ and either the spin today (top panel) or the spin at the initial conditions (bottom panel). The continuous lines represent the linear TTT growth stopped at maximum expansion ($t = 0.5t_0$). The dotted lines mark the average value at t_0.

Evolution of thew spin parameter and mass of the major progenitor of a Milky-Way-size dark matter halo.

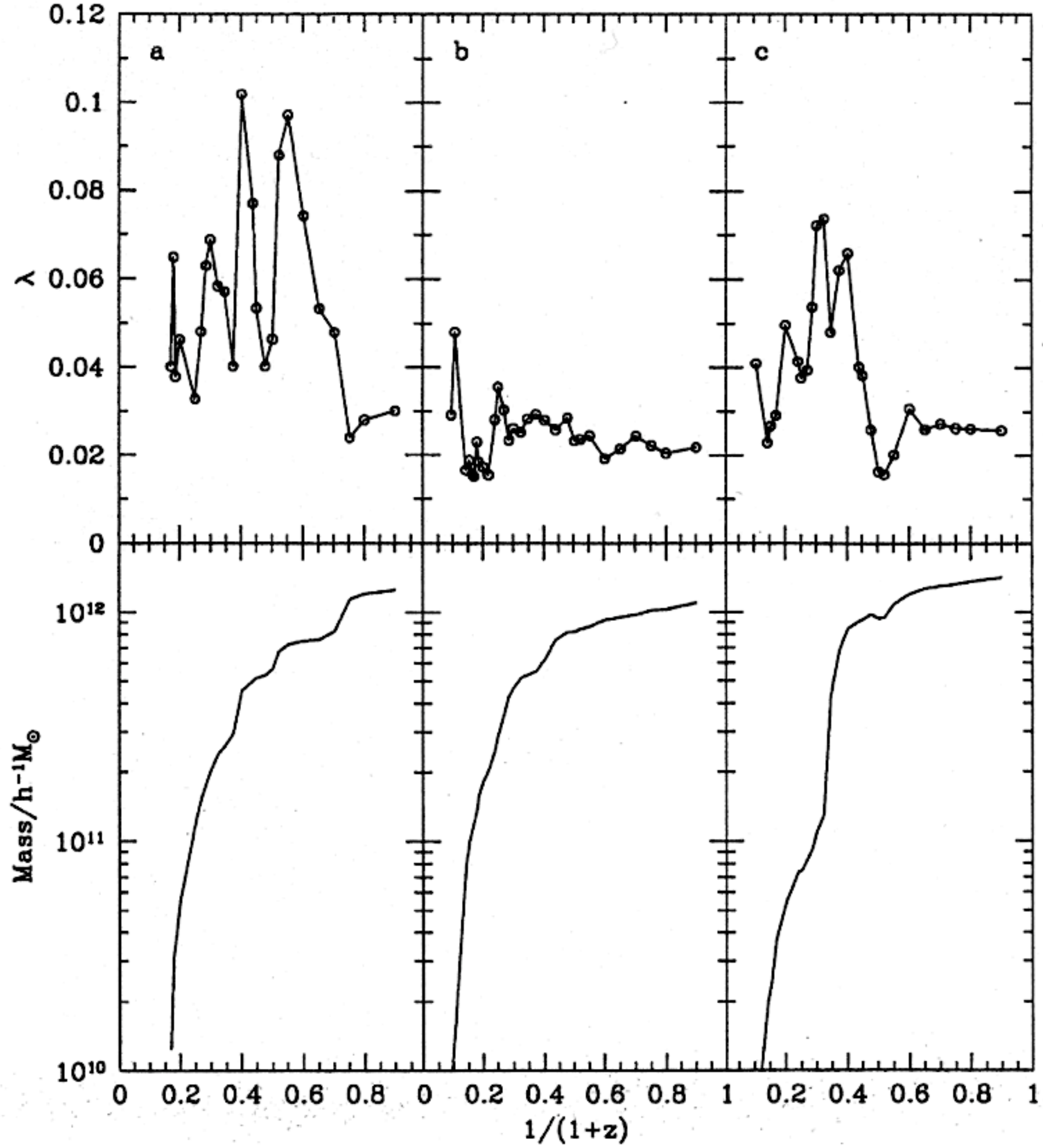


FIG. 1.— Three examples of evolution tracks of galaxy-size halos in N-body simulations. All halos show fast mass growth at high redshifts. At that epoch their spin parameters behaved very violently, but subsequently they mostly declined as the halo masses grew.

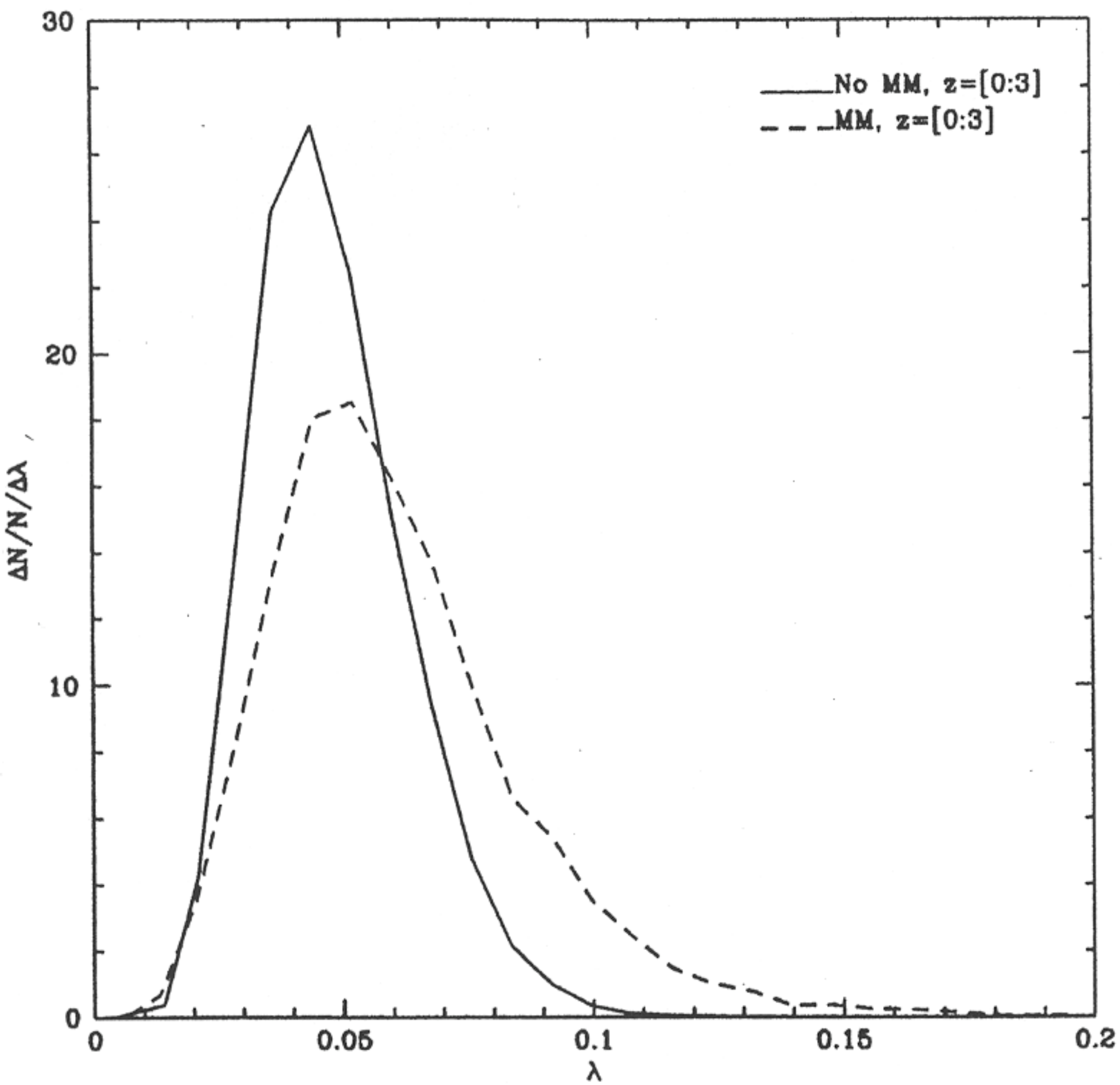


FIG. 10.— The distribution of the spin parameter for dark matter halos that had (dashed line) and didn't have (solid line) a major merger event since redshift $z = 3$. Final mass of the main progenitor at $z = 0$ is $10^{12} M_{\odot}$. The halos that had major mergers during the last ≈ 11 Gyrs on average have larger (by 25%) spin parameter than those which did not experience a major merger.