Particle Motion in axisymmetric gravitation potential

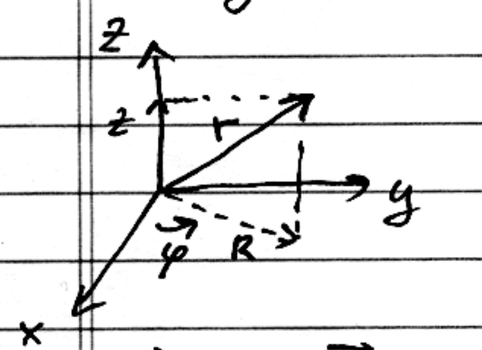
Orbits of stars in axisymmetric potentials

⇒ equations of motion
- zero-velocity curve
- types of orbits
- surfaces of section

⇒ Epicycle approximation

(I) Equations of motion and effective potential

System to study: Gravitational potential $U(\vec{r})$ does not depend on angle φ and it is symmetric relative to the plane $z=0$



Equations of motion are: $\frac{d^2\vec{r}}{dt^2} = -\nabla U(R,z)$

$\vec{e}_r$ and $\vec{e}_z$ are unit vectors along R and z axes

Radius-vector of a particle can be written as

(*) $$\vec{r} = R\vec{e}_r + z\vec{e}_z \qquad (\text{no } \vec{e}_\varphi \text{ component})$$

Gradient of force:

(**) $$\nabla U = \frac{\partial U}{\partial R}\vec{e}_r + \frac{\partial U}{\partial z}\vec{e}_z$$

as before, we have: $\dot{\vec{e}}_r = \dot{\varphi}\vec{e}_\varphi$; $\dot{\vec{e}}_\varphi = -\dot{\varphi}\vec{e}_r$; $\dot{\vec{e}}_z = 0$

Differentiate eq(*) twice with time

$$\dot{\vec{r}} = \dot{R}\vec{e}_r + \dot{z}\vec{e}_z + R\dot{\varphi}\vec{e}_\varphi$$

$$\ddot{\vec{r}} = \ddot{R}\vec{e}_r + \ddot{z}\vec{e}_z + \dot{R}\dot{\varphi}\vec{e}_\varphi + \dot{R}\dot{\varphi}\vec{e}_\varphi - R\dot{\varphi}^2\vec{e}_r + R\ddot{\varphi}\vec{e}_\varphi$$

Now, collect the terms and split the equations of motion into different components. We get three equations

$$\begin{cases} \ddot{R} - R\dot{\varphi}^2 = -\frac{\partial U}{\partial R} \\ 2\dot{R}\dot{\varphi} + R\ddot{\varphi} = 0 \\ \ddot{z} = -\frac{\partial U}{\partial z} \end{cases}$$

The second equation is a full derivate:

$$\frac{d}{dt}(R^2\dot{\varphi}) = 0 \Rightarrow L_z = R^2\dot{\varphi} = const$$

This means that z-component of the angular momentum is preserved. We have only two equations left.

Effective potential is introduced as:

$$U_{eff}(R,z) \equiv U(R,z) + \frac{L_z^2}{2R^2}$$

We can re-write the equations of motion in the following way

$$\ddot{R} = -\frac{\partial U_{eff}}{\partial R}; \quad \ddot{z} = -\frac{\partial U_{eff}}{\partial z}$$

Because the effective potential does not explicitly depend on time, the energy of each particle E is preserved

$$E = \frac{1}{2}(\dot{R}^2 + \dot{z}^2) + U_{eff}(R,z) = const$$

This defines zero-velocity curve ($\dot{R} = \dot{z} = 0$):

$$E = U_{eff}(R,z)$$

Trajectory of a particle must stay inside the curve

Types of orbits:

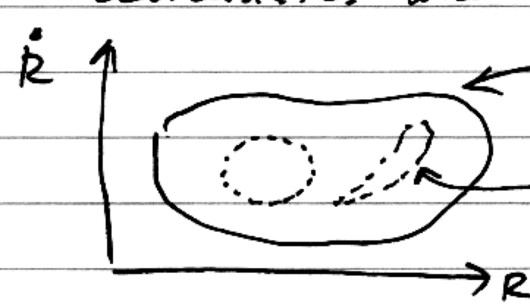
⇒ Chaotic. – Every trajectory, which has the same energy E and L_z fills all allowed phase space
– any two trajectories with the same E and L_z (but with different initial conditions) may come infinitely close one to another

⇒ Regular – Every trajectory fills only a 2d manifold in allowed 3d space.
– Trajectories on different manifolds never come close one to the other

Surface of section: reduce the phase-space dimension by one to make analysis of trajectories more clear.

This is done by looking at trajectory when it crosses a surface defined by some algebraic condition $f(R, z, \dot{R}, \dot{z}) = const$

For example, we can use: $z = 0$. What are other coordinates when a trajectory crosses $z=0$ plane?



trajectory crosses $z=0$ plane only with those R and $\dot{R}$. The velocity in z direction is defined by energy conservation $E = const$

=> Two types of regular orbits:

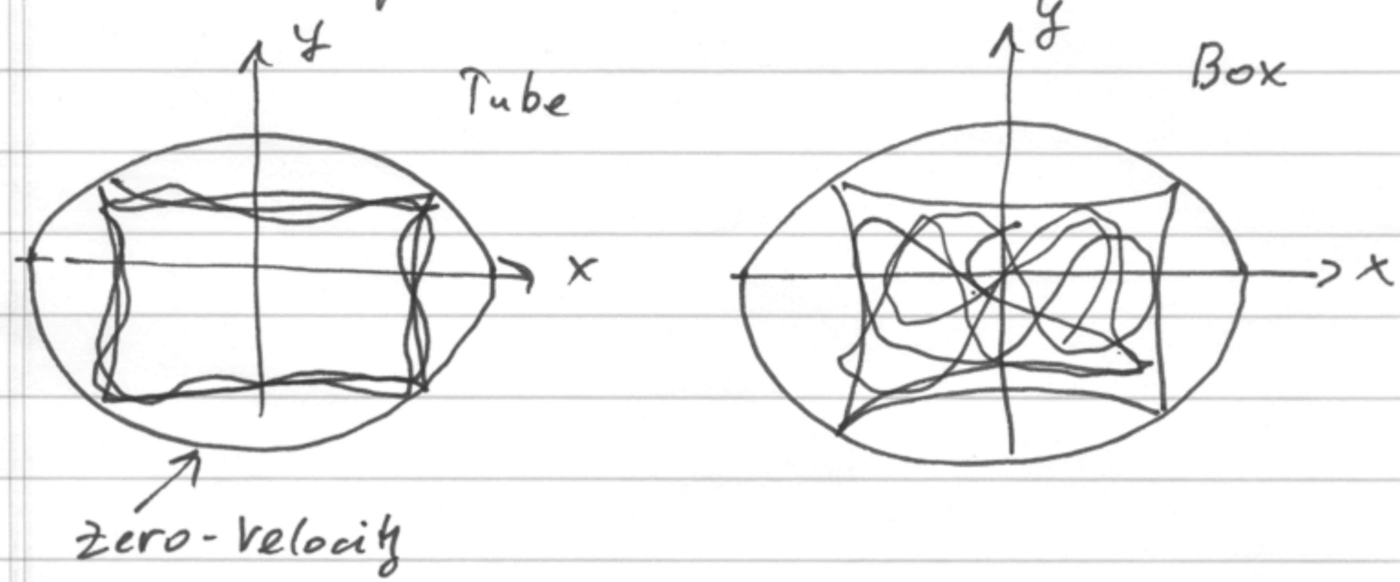
a) "tubes"

b) "boxes

Tubes do not come through R=0 point

Boxes may come infinetly close to R=0

In real space



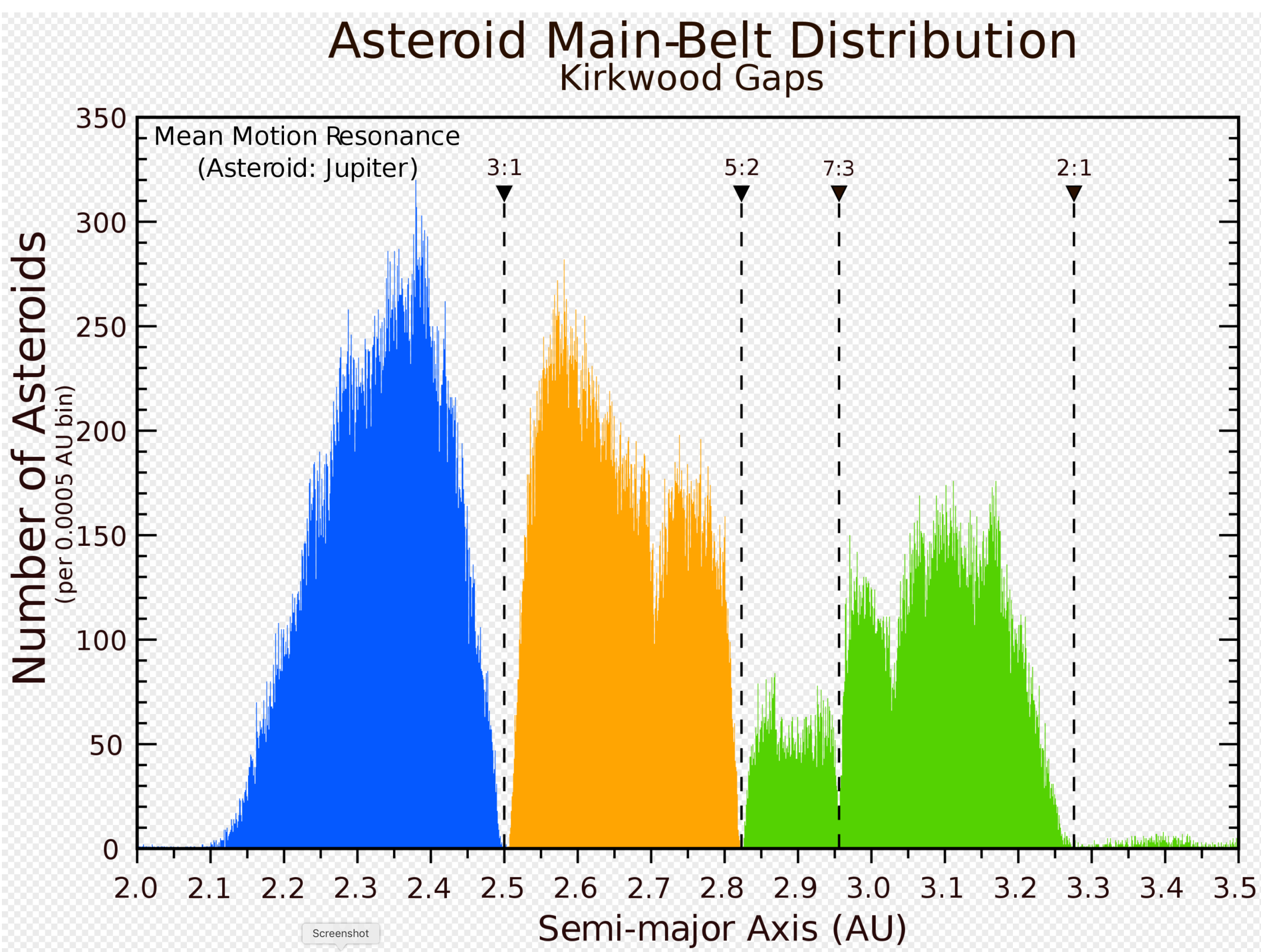
Particle Motion in non-axisymmetric gravitation potential

Stationary nondissipative systems

Examples:

resonances in barred galaxies,
planetary resonances

Example of scattering resonances: asteroid belt



Example of trapping resonances:
barred galaxies

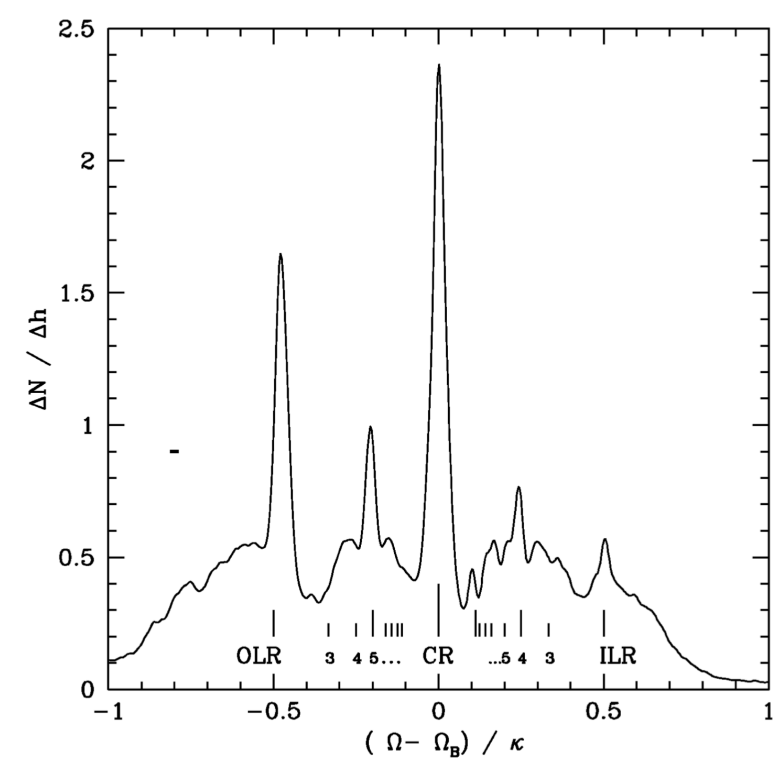


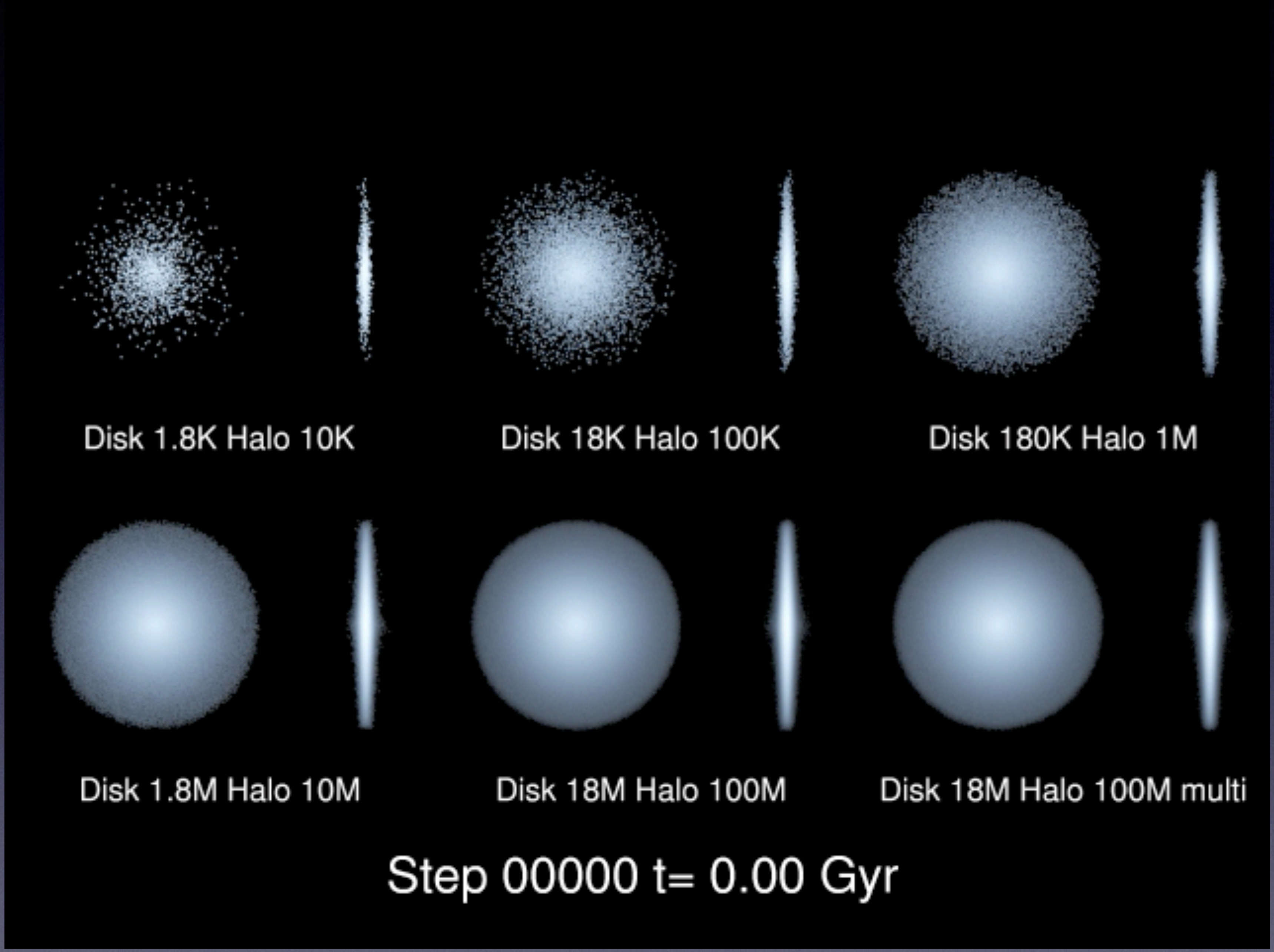
Figure 6. Distribution of the ratio $(\Omega - \Omega_B)/\kappa$ for D_{hs} for a period of 1 Gyr. The vertical axis shows the fraction of particles per unit of bin in the frequency ratio. Vertical lines represent low order resonances (±1:m) and CR. The peaks show a strong indication of trapping resonances. The

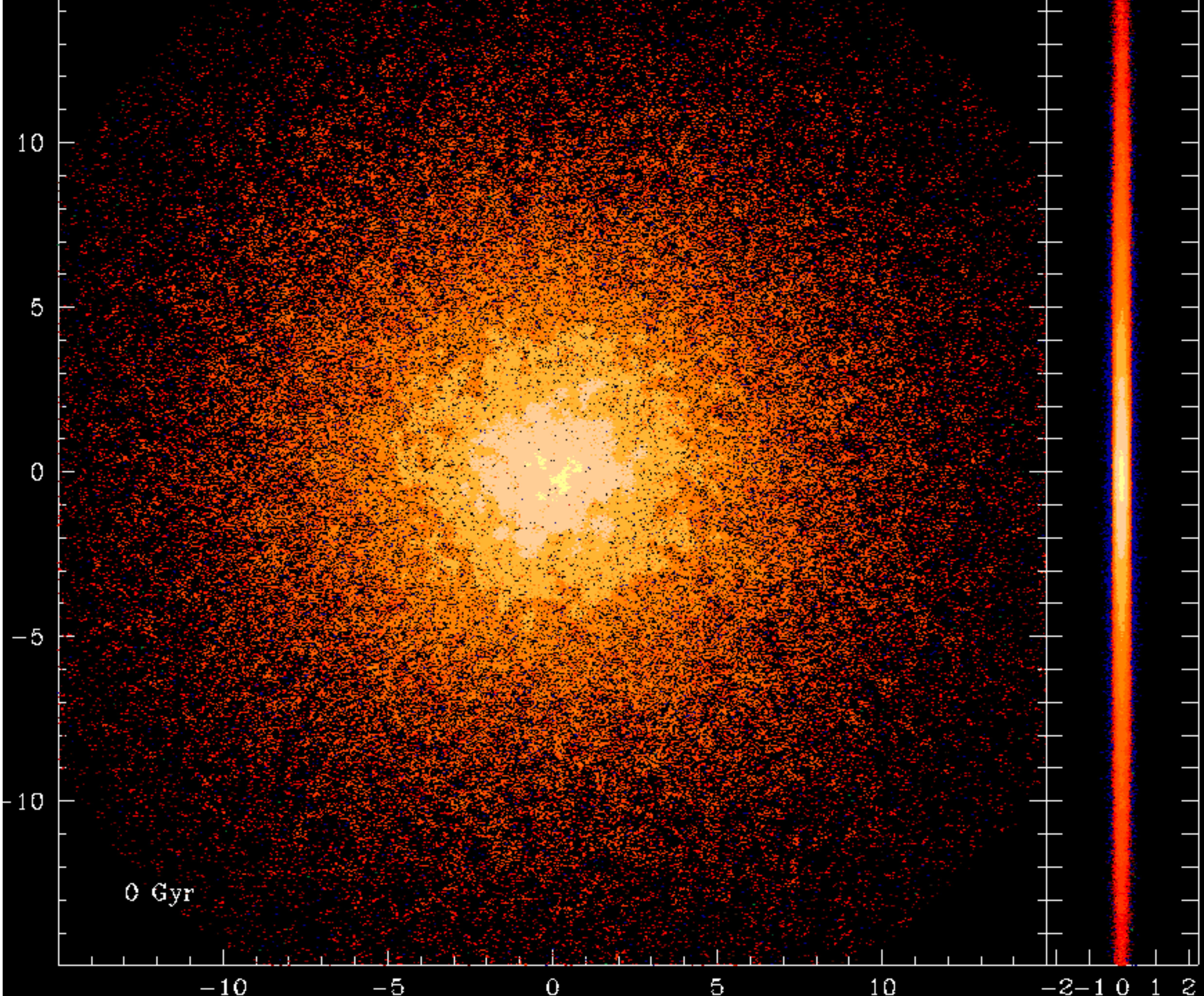
Barred galaxies can not be modeled as nearly axisymmetric systems because the dynamics of these galaxies is dominated by a strong bar which rotates around the center.

The bar interacts with galactic material and distorts galactic orbits. In particular, some galactic orbits experience dynamical resonances with the bar.

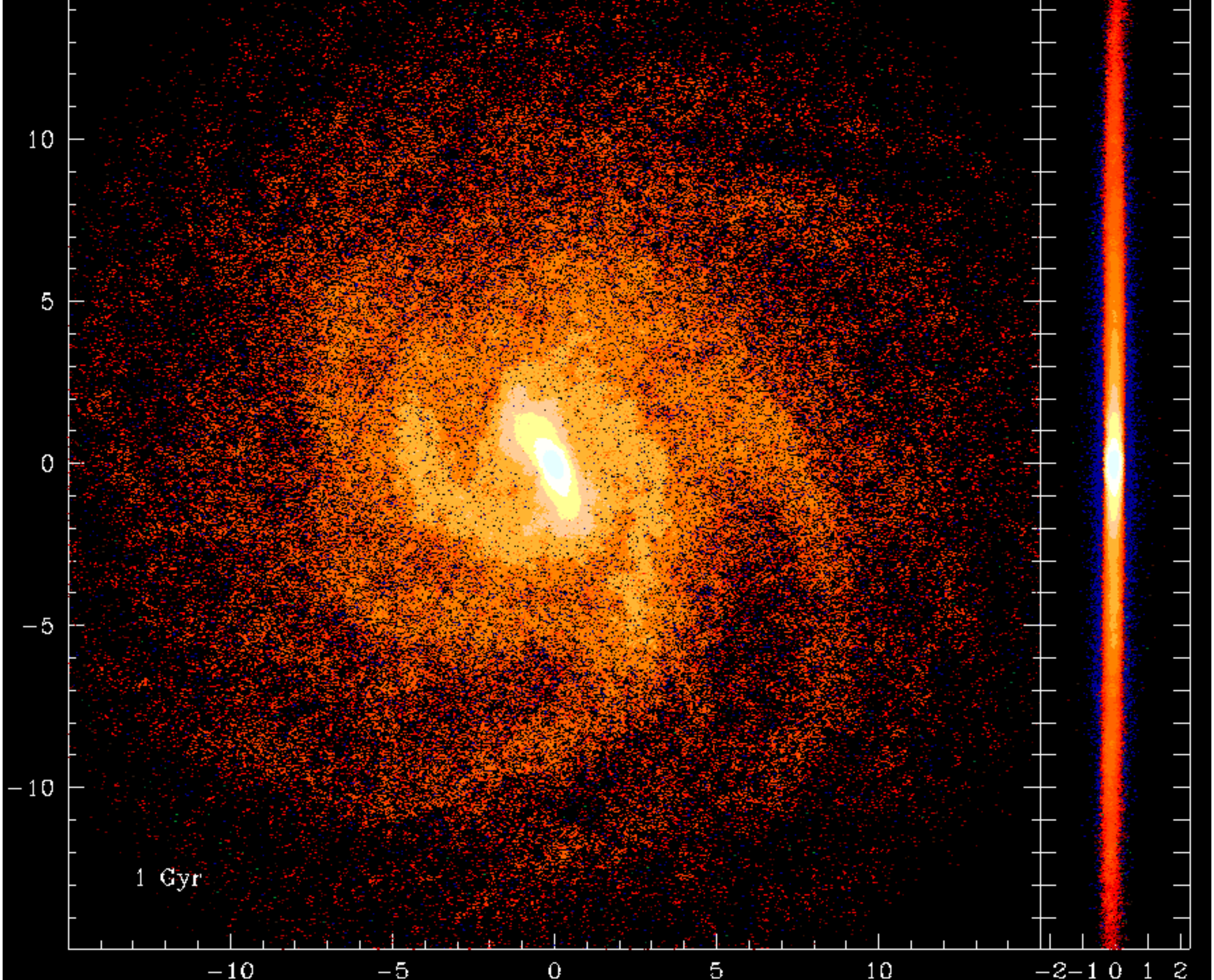
The motion in these orbits is coupled with the rotation of the bar: resonant orbits are closed orbits in the reference frame which rotates with the bar. In this frame, the bar is stationary and a resonant orbit can periodically reach the same position with respect to the bar.

Barred Galaxies: examples

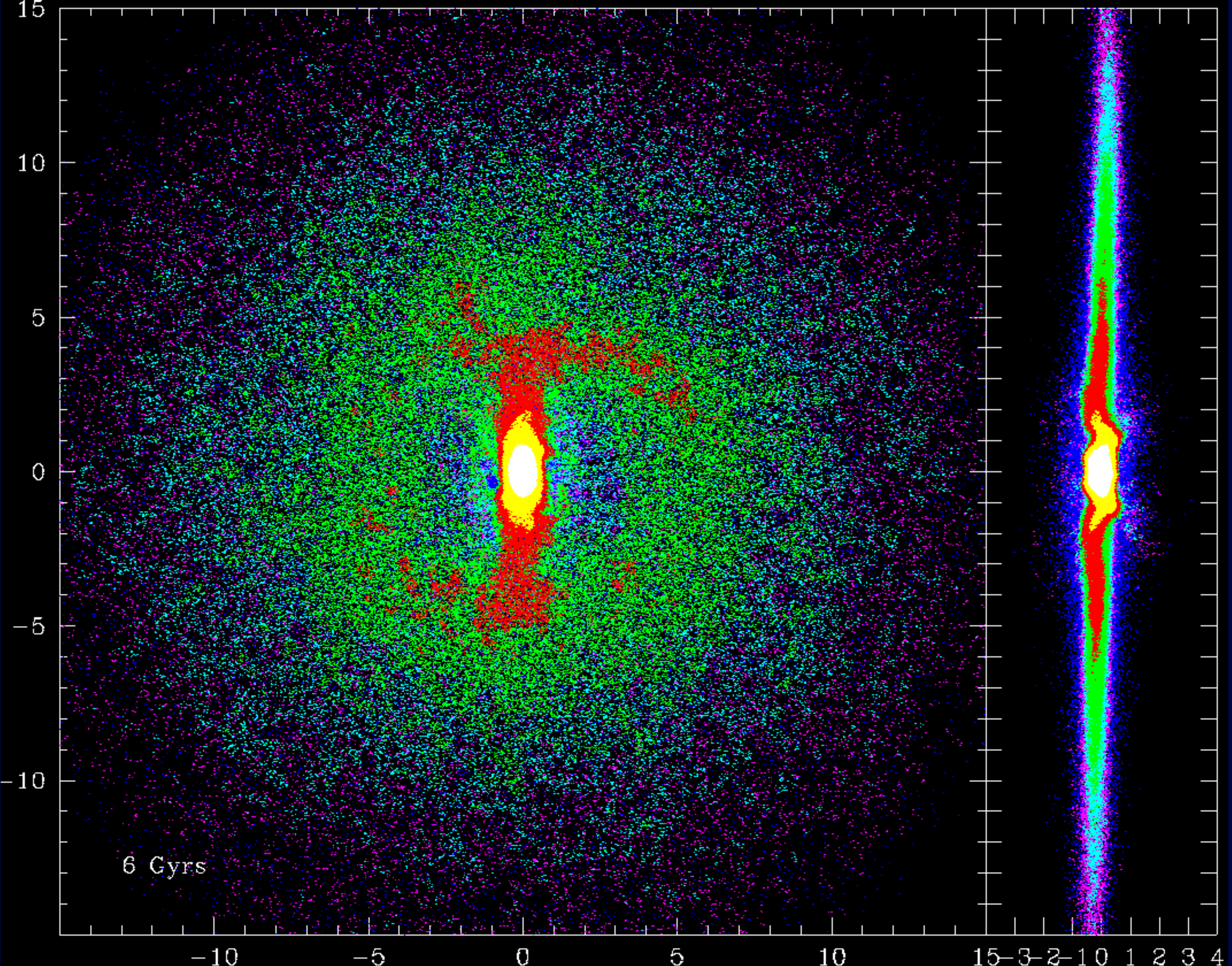

Disk 1.8K Halo 10K
Disk 18K Halo 100K
Disk 180K Halo 1M
Disk 1.8M Halo 10M
Disk 18M Halo 100M
Disk 18M Halo 100M multi
Step 00000 t= 0.00 Gyr



0 Gyr
10
5
0
−5
−10
−10
−5
0
5
10
−2
−1
0
1
2



10
5
0
−5
−10
1 Gyr
−10
−5
0
5
10
−2−1 0 1 2



15
10
5
0
−5
−10
6 Gyrs
−10
−5
0
5
10
15
−3
−2
−1
0
1
2
3
4

What resonances do and what they do not

Simple expectations

$$\ddot{y} = -\omega_0^2 y + \alpha \cos\omega t \Rightarrow y = \frac{\alpha}{(\omega_0^2 - \omega^2)} \cos\omega t$$

$$\omega = \omega_0 \Rightarrow y \propto \alpha t \sin\omega t$$

Forced 1d pendulum

`Perturbation expansion`

$$A \propto \frac{1}{\vec{m}\cdot\vec{\Omega} - q\Omega_{bar}}$$

This is valid only in 1-dimensional case: not true in 2- or 3-dimensions frame which rotates with the bar. In this frame, the bar is stationary and a resonant orbit can periodically reach the same position with respect to the bar. A resonant orbit is therefore a periodic orbit in this reference frame and its dynamical frequencies are commensurable.

In general, these oscillations could be described by three instantaneous orbital frequencies:

- radial frequency $\varkappa$,
- vertical frequency ν
- angular frequency Ω.

The angular frequency of the rotation of the bar is Ω_B

relationship of commensurability: We mostly will be interested in cases with motion close to the galactic plane: So, the resonant condition is reduced to

$$l\varkappa + m(\Omega - \Omega_B) = 0$$

Name	l	m	n	$\frac{\Omega - \Omega_B}{\kappa}$
CR	0	1	0	0
ILR	-1	2	0	0.5
OLR	1	2	0	-0.5
UHR	-1	4	0	-0.25

CR = corotation resonance (angular orbital frequency is equal to frequency of the bar; analog of Trojan asteroids in the solar system)
ILR = inner Lindblad resonance (orbits inside co-rotation radius, for every orbital period there are two radial periods)
OLR = outer Lindblad resonance (orbits outside co-rotation radius)

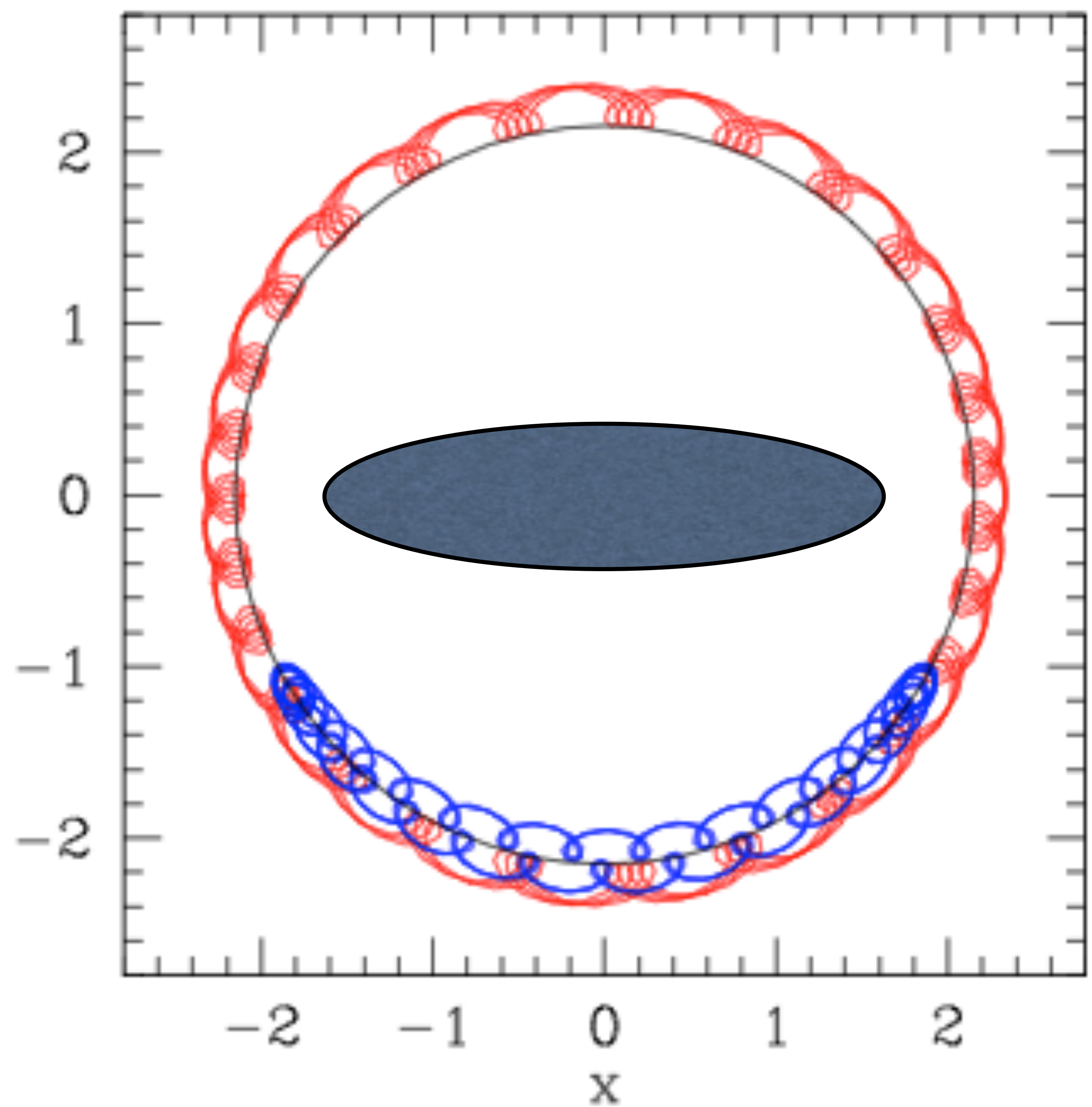
$$m\Omega_r + n\Omega_\varphi + k\Omega_z = q\,\Omega_{bar}$$

Theorem: orbits on exact resonances do not experience any net torque or net change of energy

anything interesting happen close to a resonance?

Orbits around corotation resonances.
Frame rotates with the bar.
Exact resonances are Lagrange points.
All other orbits oscillate along radius (fast) and librate (slow) in tangential direction.

No net change in energy of ang.momentum once averaged over an orbit or over a mixed population of orbits



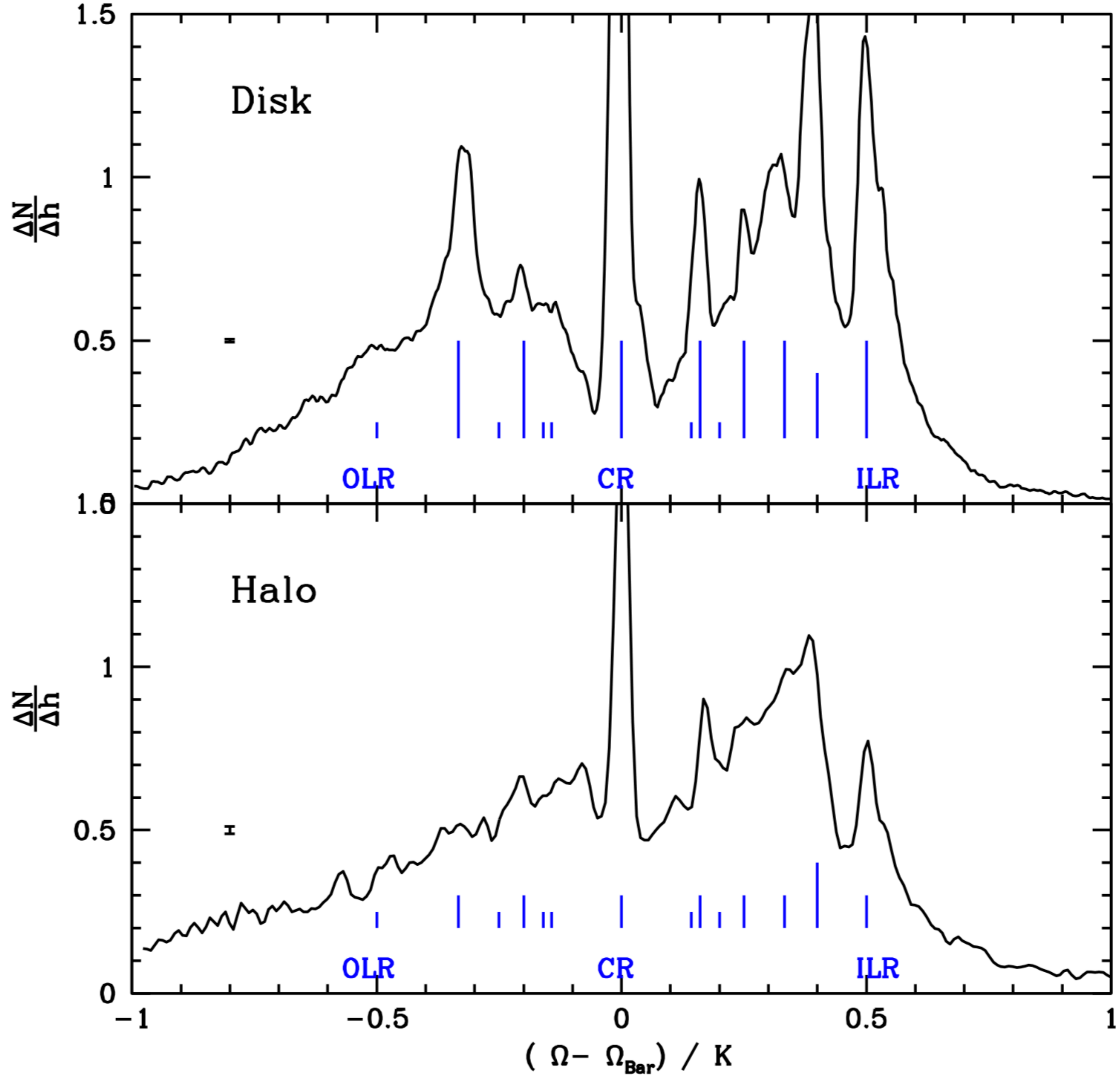


Figure 22. Bottom: Distribution of the ratio $(\Omega - \Omega_B)/\kappa$ for particles in the halo chosen to stay close to the disk of model 1. The lines present different resonances. The corotation and the inner Linblad resonances are clearly present in the halo. Top: the same for the disk of model 1. The errorbars in both plots are the 1σ error using poison noise.

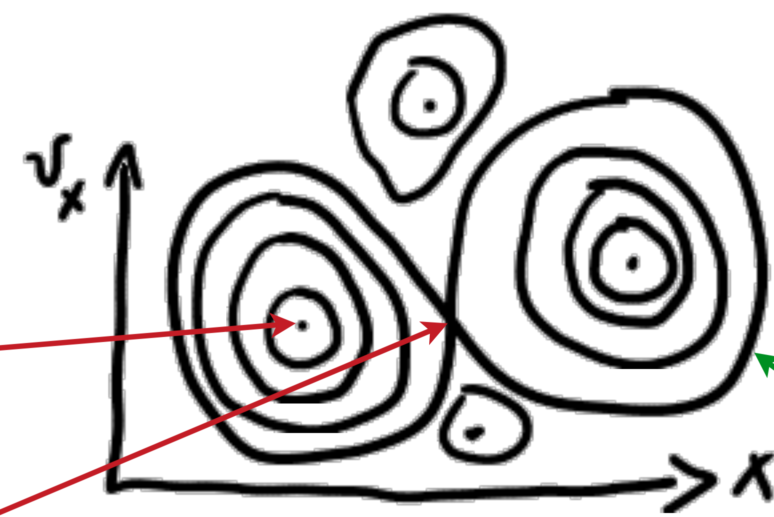
Resonances: few notions

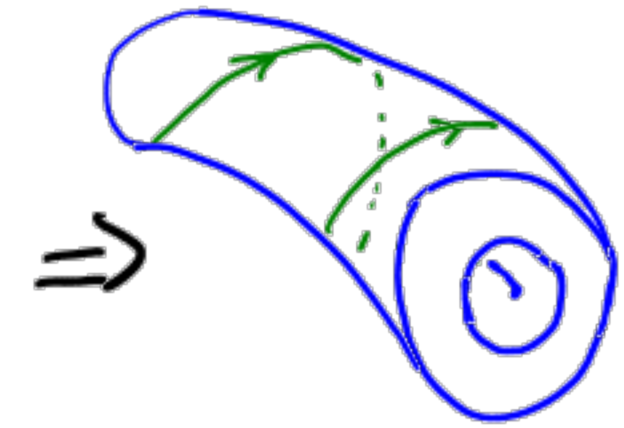
$$m\Omega_r + n\Omega_\varphi + k\Omega_z = q\,\Omega_{bar}$$

$\Omega_r \quad \Omega_\varphi \quad \Omega_z$ ← orbital frequencies

Ω_{bar} bar frequency

here m, n, k, and q are integers



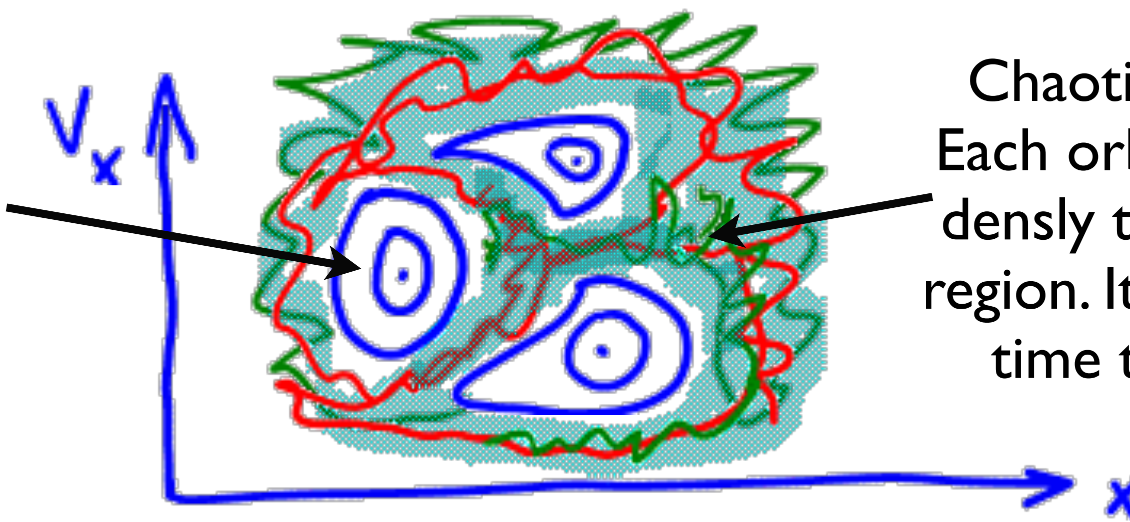


Elliptical Resonance: closed orbit in phase-space

Hyperbilic Resonance

Separatrix or homoclinic orbit

Regular orbits in a domain of this prime resonance. Averaged over time frequencies of these orbits are the same as the frequency of the resonance

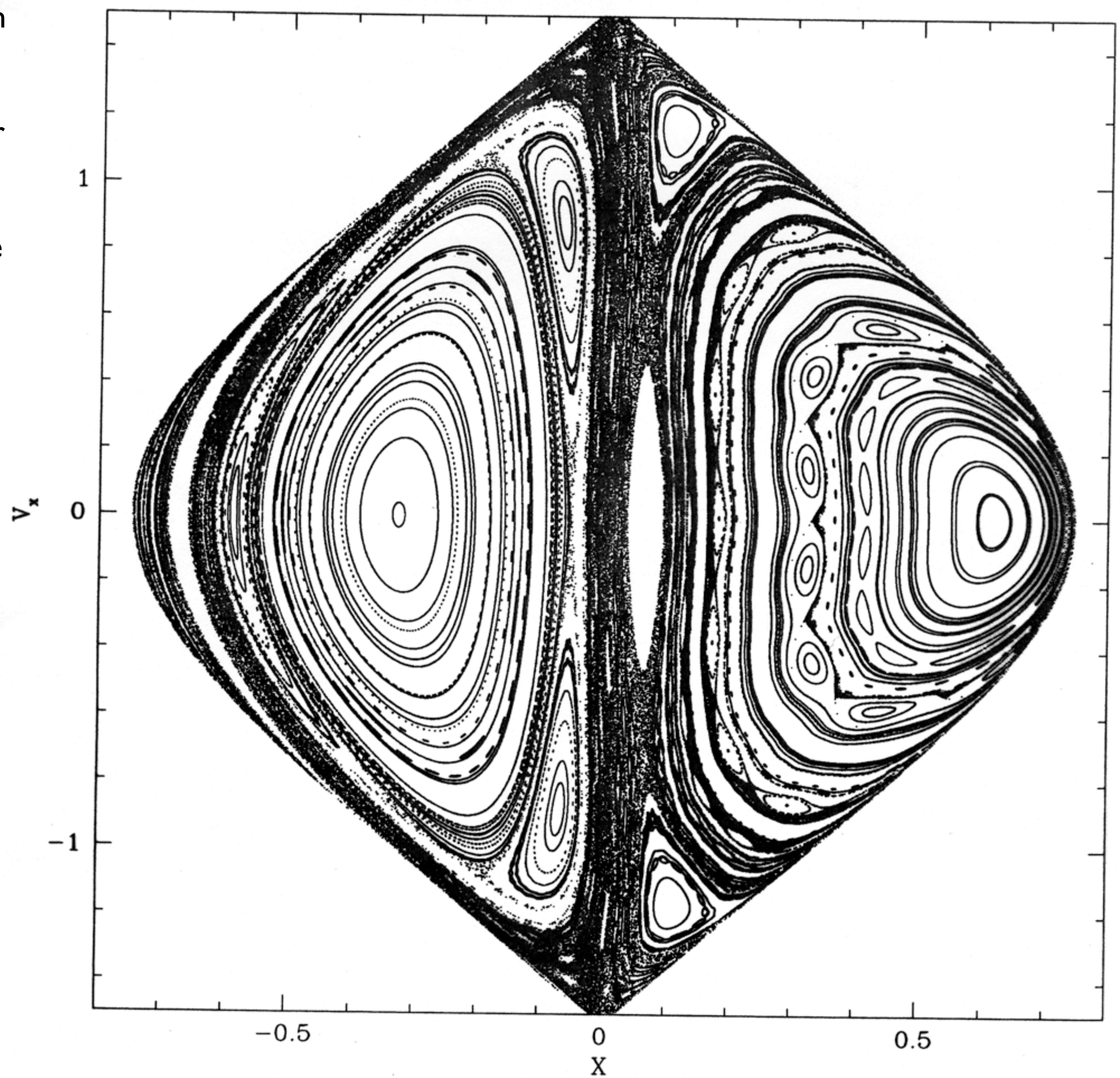


Chaotic orbits. Each orbit covers densly the whole region. It takes inft time to do it.

Transition between region of a resonance and domain of chaotic orbits. Secondary resonances get bigger. Areas of chaotic orbits appear between regular orbits.

An example of surface of section in a realistic gravitational potential of disk+halo+bar system. All orbits are in the plane of the disk. The bar rotates with a constant pattern speed and the reference frame is chosen to rotate together with the bar.

All orbits were selected to have the same energy. They have different initial coordinates. When an orbit crosses y=0 plane, its (x,Vx) coordinates are recorded if its Vy>0. After a long period of time all recorded pairs of point (x,Vx) are plotted.



Types of orbits:

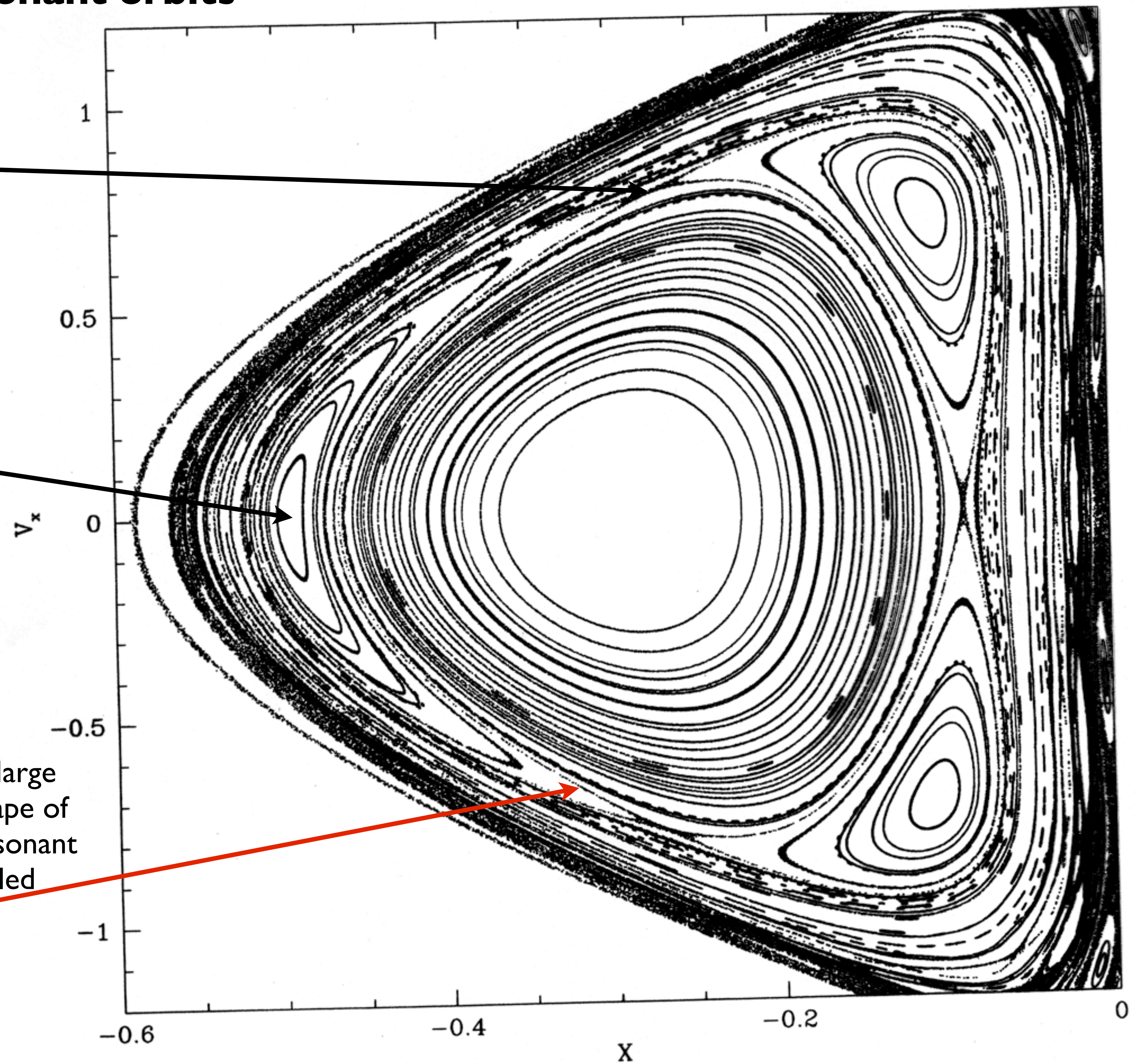
- resonant or closed orbits are those, which cross the ‘bulls eyes’: centers of ellipses in the plot or at intersections of separatrixes
- regular orbits, which produce closed loops on the plot
- irregular orbits, which populate grey regions

Two types of resonant orbits

Hyperbolic resonant orbit: it is an unstable point

Elliptical resonant orbit: it is a stable point

Close look at the domain of large resonant orbits. Note the shape of orbits, which separate the resonant domains. Those orbits are called 'separatrixes'



Zoom-in on the region of transition from a domain of regular orbits to irregular (or chaotic) orbits